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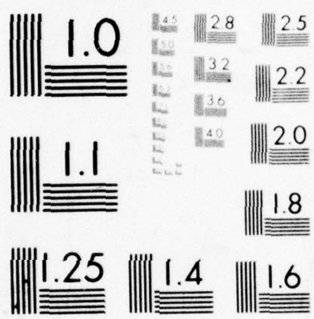

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CHARACTERIZATIONS OF SUMS OF DYADS
AND OF KRONECKER PRODUCTS

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We characterize those transformations $\{U_i\}$, $\{V_i\}$ where $\Sigma_i(U_i][V_i) = R$, where R is given in advance. Related characterizations are given for sums of the form $\Sigma_i U_i \otimes V_i$.

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INTRODUCTION

In a previous paper [2] we studied the (decomposable) Kronecker product $A \otimes B$ of finite rank linear transformations A and B on complex Hilbert space, and the relation to (decomposable) dyadic products $(U][V)$, where U and V are also finite rank linear transformations. (Definitions of these products are in the following section.) In particular, we characterized those <u>linearly independent</u> families $\{U_i\}$, $\{V_i\}$ with the property that

$$\sum_{i=1}^{N} (U_i][V_i) = A \otimes B^t \tag{1.1}$$

for given transformations A and B. (B^t denotes the transpose of B.) In this paper, we extend the analysis of (1.1) by characterizing the generating sums

$$\sum_{i=1}^{N} (U_i][V_i) = \underset{\sim}{R} \tag{1.2a}$$

$$\sum_{i=1}^{N} (U_i \otimes V_i) = \underset{\sim}{S} \tag{1.2b}$$

where $\underset{\sim}{R}$ and $\underset{\sim}{S}$ are general linear transformations and not just decomposable Kronecker of dyad products. Thus, this paper improves upon [2] in the three respects:

(a) The sums (1.2a) and (1.2b) are more general. Moreover, they will be characterized without assuming linear independence of the families

$\{U_i\}$, $\{V_i\}$. (Th. 3.2.)

(b) The relations among the transformations $\underset{\sim}{R}$, $\underset{\sim}{S}$, $\{U_i\}$, $\{V_i\}$ will be natural in the sense that they do not depend on particular matrix representations of the functions in question. This matrix dependence was used in [2] in the analysis of (1.1).

(c) We show a duality between the operations "][" and "⊗" , so that characterizing (1.2a) is equivalent to characterizing (1.2b). (Th. 4.2, 4.3.)

We briefly mention one motivation for studying equations of the form (1.1). If $A = (a_{ij})$ and $B = (b_{jk})$ are each 2×2 scalar-entried matrices, then the classical definition of matrix multiplication says that the four entries

$$c_{ik} = \sum_{j=1}^{2} (a_{ij} \cdot b_{jk}) \ , \quad i \ , \ k = 1 \ , \ 2 \ , \qquad (1.3)$$

of the matrix product AB generally require eight scalar multiplications, viz., the eight products $(a_{ij} \cdot b_{jk})$, i , j , $k = 1$, 2 . In the landmark paper of Strassen [5], he shows that there are seven scalar products, $(a_{11} + a_{22}) \cdot (b_{11} + b_{12})$, $(a_{11} + a_{12}) \cdot (b_{22})$, etc., whose sums and differences suffice to produce the four scalars c_{ik} of (1.3). Now in [3] it is shown that if A and B are consistent rectangular matrices (i.e., AB is defined), and if I is an identity matrix, then the scalars c_{ik} of AB each identify with a certain Kronecker product $E_{ik} \otimes I$, while the generating scalar products (such as $(a_{11} + a_{22}) \cdot (b_{11} + b_{12})$, $(a_{11} + a_{12}) \cdot (b_{22})$,...) each identify with a unique dyadic product (U][V) . To minimize the number of scalar products which produce all entries of matrix AB is equavlent, therefore, to minimizing the number of dyadic products $(U_i][V_i)$ whose sums and differences produce all the Kronecker products $E_{ik} \otimes I = \sum \pm (U_i][V_i)$. From (1.2a) and (1.2b) we derive some interesting special cases.

PRELIMINARY RESULTS AND DEFINITIONS

Families $\{u_i\}$, $\{v_i\}$, $i = 1$, 2 ,..., N in Hilbert space are biorthonormal if $\langle u_i,v_j\rangle = \delta_{ij}$, the Kronecker delta. $\mathcal{L}(H,K)$ denotes all bounded linear transformations sending Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{K}$; among these are the rank one dyads $(x \times y)$ for $x \in K$, $y \in H$, where for all $z \in \mathcal{H}$, $(x \times y):z \to \langle z,y\rangle x$. The adjoint of $A \in \mathcal{L}(H,K)$ is A^* (which belongs to $\mathcal{L}(K,H)$) and is defined by $\langle Ay,x\rangle = \langle y,A^*x\rangle$ for all $x \in K$, $y \in H$. From this we see that for $(x \times y) \in \mathcal{L}(H,K)$, $A \in \mathcal{L}(K,H')$ and $B \in \mathcal{L}(H',H)$, we have

$$A(x \times y) = (Ax \times y) ,$$

and (2.1)

$$(x \times y)B = (x \times B^*y) .$$

The space $\mathcal{L}(H,K)$ accepts an inner product $[\cdot,\cdot]$ defined by $[A,B] = tr(B^*A)$, the trace of B^*A in $\mathcal{L}(H,H)$, where A , $B \in \mathcal{L}(H,K)$. The dyad of transformations $A \in \mathcal{L}(H_1,H_2)$ and $B \in \mathcal{L}(K_1,K_2)$ is denoted by $(A][B)$, where

$$(A][B):C \to [C,B]A \tag{2.2}$$

for all $C \in \mathcal{L}(K_1,K_2)$. cf. [4, Ch. 5, §5].

For Hilbert space H , $\overline{H}$ denotes the Hilbert space of linear functionals on H , where for $x \in H$, $\overline{x} \in \overline{H}$ is given by

$$\overline{x}:y \to \langle y,x\rangle \tag{2.3}$$

for all $y \in \mathcal{H}$. The transpose A^t of $A \in \mathcal{L}(H,K)$ is that transformation sending $\overline{K}$ to $\overline{H}$ defined by $A^t(\overline{y})(x) = \overline{y}(Ax)$ for all $x \in H$, $\overline{y} \in \overline{K}$. As a special case the transpose of a dyad is $(x \times y)^t = (\overline{y} \times \overline{x})$ while its adjoint is $(x \times y)^*$ $= (y \times x)$. If $A \in \mathcal{L}(H_1,H_2)$, $B \in \mathcal{L}(K_1,K_2)$, then the Kronecker product $A \otimes B^t$ of transformations A and B^t is given by

$$A \otimes B^t : C \to ACB \tag{2.4}$$

for all $C \in \mathcal{L}(K_2,H_1)$. cf. [4, p. 211].

For any transformation A , the symbols rgA and n(A) denote the range of A and the nullspace of A , respectively, while $M^\perp$ denotes the orthogonal complement of subsapce M . If $M \subset H$, then

$$A \in \mathcal{L}(M,\mathrm{rg}A) \subset \mathcal{L}(H,H') \tag{2.5}$$

will mean $A : M \to \mathrm{rg}A \subset H'$, $M \subset H$, and

$$A : M^\perp \to 0 \text{ , (i.e., } M \text{ is orthogonal to } n(A)) \text{ .}$$

An important relationship between the dyad and Kronecker products is given by

$$(x \times y)][(u \times v) = (x \times u) \otimes (\overline{y} \times \overline{v}) \tag{2.6}$$

for all x , y , u , v in (possibly different) Hilbert spaces. cf. [1, p. 131]. Finally, for vectors $\{X_i : i = 1 , 2 , \ldots , N\}$ the symbol $sp\langle X_i : i=1 , 2 , \ldots , N\rangle$ or $sp\langle X_i\rangle$ will denote their linear span.

THE MAIN RESULT

We now characterize the terms $(U_i][V_i)$ of sum (1.2a) given transformation $\underset{\sim}{R}$. The following terminology will be convenient

Definition 3.1. For the families of linear transformations $\{U_i\} \subset \mathcal{L}(H_1,H_2)$, $\{V_i\} \subset \mathcal{L}(K_1,K_2)$, we define their respective ranks $\underset{\sim}{r_u}$, $\underset{\sim}{r_v}$ by

$$\begin{aligned} r_u &= \text{dimension } (sp\langle U_i\rangle) , \\ r_v &= \text{dimension } (sp\langle V_i\rangle) . \end{aligned} \tag{3.1}$$

For any linearly independent family $\{U_i\}$, then biorthonormal complement to $\{U_i\}$ is that unique linearly independent subset $\{\overline{U}_i\}$ of $sp\langle U_i\rangle$ such that

$$[\hat{U}_i,U_j] = S_{ij} , \text{ the Kronecker delta.} \tag{3.2}$$

Remark. Biorthonormality is a natural generalization of orthonormality. Indeed, if $\{U_i\}$ is not only linearly independent, but orthonormal as well, then we would have $\hat{U}_i = U_i$ for each i. Also suggestive of orthonormality is that any element $Z \epsilon sp\langle U_i\rangle = sp\langle \hat{U}_i\rangle$ has representation

$$Z = \sum_i [Z,\hat{U}_i]U_i = \sum_i [Z,U_i]\hat{U}_i \tag{3.3}$$

which follows from (3.2).

Theorem 3.2. (1) Given any linear transformation

$$\underset{\sim}{R}:\mathcal{L}(K_1,K_2) \to \mathcal{L}(H_1,H_2)$$

(2) Given integer $k \geq 0$, and any (r_u+k)-element set of transformations $\{U_i\}$, such that $\operatorname{rg}\underset{\sim}{R} \subset \operatorname{sp}\langle U_i \rangle \subset \mathcal{L}(H_1,H_2)$, where (by reordering if necessary) the first r_u elements form a basis for $\operatorname{sp}\langle U_i \rangle$. Accordingly, if $k > 0$ we are given unique scalars $\{\alpha_i^{(j)}\}$, defined by

$$U_j = \sum_{i=1}^{r_u} \alpha_i^{(j)} U_i \ , \quad j = r_u + 1 \ , \ r_u + 2 \ , \ldots , \ r_u + k \ .$$

(3) Given arbitrary linear transformations (for $k > 0$) ,

$$V_{r_u+1} , V_{r_u+2} , \ldots , V_{r_u+k} \in \mathcal{L}(K_1,K_2) \ .$$

We conclude that

$$\sum_{i=1}^{N} (U_i][(V_i) = \underset{\sim}{R} \ , \quad N = r_u + k \qquad (3.4)$$

if, and only if, each $V_i = \mathcal{L}(K_1,K_2)$, $i = 1 , 2 , \ldots , r_u$, is given by

$$V_i = \underset{\sim}{R}^*(\hat{U}_i) - \sum_{j=r_u+1}^{N} \overline{\alpha}_i^{(j)} V_j \ , \ i = 1 \ , \ 2 \ , \ldots , \ r_u \qquad (3.5)$$

where $N = r_u + k$, and the scalars $\{\alpha_i^{(j)}\}$ are defined in Hypothesis (2). If $k = 0$, then $V_i = \underset{\sim}{R}^*(\hat{U}_i)$

<u>Proof (3.4)⇒(3.5)</u>. We apply both sides of (3.4) to arbitrary $Z \in \mathcal{L}(K_1,K_2)$ to obtain

$$\sum_{i=1}^{N} [Z,V_i]U_i = \underset{\sim}{R}(Z) \in \operatorname{rg}\underset{\sim}{R} \ . \qquad (3.6)$$

That is, $\underset{\sim}{R}(Z) \in sp\langle U_i \rangle$, so that from (3.3), we may write

$$\begin{aligned} R(Z) &= \sum_{i=1}^{r_u} [\underset{\sim}{R}(Z), \hat{U}_i] U_i \\ &= \sum_{i=1}^{r_u} [Z, \underset{\sim}{R}^*(\hat{U}_i)] U_i \end{aligned} \tag{3.7}$$

We reformulate (3.6) as follows, where $N = r_u + k$:

For all $Z \in \mathcal{L}(K_1, K_2)$,

$$\sum_{i=1}^{r_u} [Z, V_i] U_i + \sum_{j=r_u+1}^{N} [Z, V_j] U_j = \underset{\sim}{R}(Z)$$

$$\rightleftarrows \sum_{i=1}^{r_u} [Z, V_i + \sum_{j=r_u+1}^{N} \bar{\alpha}_i^{(j)} V_j - \underset{\sim}{R}^*(\hat{U}_i)] U_i = 0 \ .$$

This last equation is obtained by substitution of equations for the U_j's (Hypothesis (2)) and (3.7) This implies that

$$V_i = \underset{\sim}{R}^*(U_i) - \sum_{j=r_u+1}^{N} \bar{\alpha}_i^{(j)} V_j \ , \quad i = 1 \ , \ 2 , \ldots , r_u \ ,$$

which establishes that (3.4)⇒(3.5).

(3.5)⇒(3.4): Substitute (3.5) into (3.4) and the proof is by verification. This ends the theorem.□

Remark. We note that our Hypotheses (2) wherein we assume $sp\langle U_i \rangle \supset rg\underset{\sim}{R}$, is also a necessary condition if (3.4) is to obtain. This is immediate from (3.6). Finally, we observe that Theorem 3.2 is the generalization of [2, Th. 3.1] which replaces $\underset{\sim}{R}$ with $A \otimes B^t$ and requires linear independence of the families $\{U_i\}$, $\{V_i\}$.

A DUALITY THEOREM

Having completely characterized compenent terms $(U_i][V_i)$ for sums (3.4), we raise the question whether a characterization is possible with terms of the form $U_i \otimes V_i$, i.e., if "$\otimes$" replaces "][" . With minor modification, (3.4) is equivalent to such a sum. In developing this idea, we come to

<u>Definition 4.1</u>. Given $U \in \mathcal{L}(H_1,H_2)$, $V \in \mathcal{L}(K_1,K_2)$. Then the linear transformation ϕ is defined by

$$\phi\ (U][V) = U \otimes \overline{V} \tag{4.1}$$

(linear extension defines ϕ on $sp\langle (U][V)\rangle$.

We present the theorem which says that in (4.1), operations "][" and "$\otimes$" may be interchanged.

<u>Theorem 4.2</u>. The equation

$$\phi\ (U][V) = U \otimes \overline{V} \tag{4.1}$$

is valid for all $U \in \mathcal{L}(H_1,H_2)$, $V \in \mathcal{L}(K_1,K_2)$ if, and only if,

$$\phi\ (P \otimes Q) = (P][\overline{Q}) \tag{4.2}$$

for all $P \in \mathcal{L}(K_2,H_2)$, $Q \in \mathcal{L}(\overline{K}_1,\overline{H}_1)$.

<u>Proof</u>. Without loss of generality (due to linear extension), we may suppose U , V , P and Q are all rank one transformations. Now set

$$U = (u_2 \times u_1) \qquad u_1 \in H_1 \ , \quad u_2 \in H_2$$
$$V = (v_2 \times v_1) \qquad v_1 \in K_1 \ , \quad v_2 \in K_2$$
$$P = (u_2 \times v_2)$$
$$Q = (\bar{u}_1 \times \bar{v}_1) \ .$$

Then

$$\phi(U][V) = U \otimes \overline{V}$$
$$\rightleftarrows \phi(\,(u_2 \times u_1)][(v_2 \times v_1)) = (u_2 \times u_1) \otimes (\bar{v}_2 \times \bar{v}_1)$$
$$\rightleftarrows \phi((u_2 \times v_2) \otimes (\bar{u}_1 \times \bar{v}_1)) = (u_2 \times v_2)][(u_1 \times v_1) \quad \text{from (2.6)}$$
$$\rightleftarrows \phi(P \otimes Q) \qquad = (P][\overline{Q}) \ .$$

Since the above equations hold for rank one transformations, linear extension guarantees validity for arbitrary finite rank transformations, and the theorem is proved.□

An immediate application of this result is a dual formulation of Theorem 3.2, which we state without proof.

<u>Theorem 4.3</u> (dual to Th. 3.2). Given Hypotheses (1), (2) and (3) of Theorem 3.2 then

$$\sum_{i=1}^{N} (U_i][(V_i) = \underset{\sim}{R}$$

if, and only if,

$$\Sigma\, U_i \otimes \overline{V}_i = \phi\ (\underset{\sim}{R})$$

where $\phi(X][Y) = X \otimes \overline{Y}$,

if, and only if,

$$V_i = \underset{\sim}{R}^{*}(\hat{U}_i) - \sum_{j=r_u+1}^{N} \bar{\alpha}_i{}^{(j)} V_j \ , \quad i = 1 \ , \ 2 \ , \ldots , \ r_u \ .$$

<u>Remark</u>. One difficulty with Theorem 4.3 is that although we may know a good deal about $\underset{\sim}{R}$, we may not be able to understand the precise form of $\phi(\underset{\sim}{R})$. In the next section, we

present applications and special cases where this difficulty does not arise.

SOME APPLICATIONS AND SPECIAL CASES

The case $\{U_i\}$ is linearly independent.

Theorem 5.1. If $\underset{\sim}{R}$ is a linear transformation of finite rank M, and if $\{U_1, U_2, \ldots, U_M\}$ is a basis for $rg\underset{\sim}{R}$, then there is one and only one basis $\{V_1, V_2, \ldots, V_M\}$ of rgR^* such that

$$\sum_{i=1}^{M} (U_i][(V_i) = \underset{\sim}{R} . \tag{5.1}$$

In fact,

$$V_i = \underset{\sim}{R}^*(\hat{U}_i) \quad i = 1 , 2 , \ldots , M . \tag{5.2}$$

Proof. This is just Theorem 3.2 when $k = 0$.

Now Theorem 5.1 gives us, as a special case, Theorem 3.4 of [2], which we state, along with a dual form which is a consequence of Theorem 4.2.

Theorem 5.2 (plus dual to [2, Th. 3.4]). Let A and B be linear transformations of ranks r and s, respectively. Let $\{U_1, U_2, \ldots, U_{rs}\}$ be a basis for $\mathcal{L}(rgB^*, rgA)$. Then there is one and only one basis $\{V_1, V_2, \ldots, V_{rs}\}$ for $\mathcal{L}(rgB, rgA^*)$ such that

$$\sum_{i=1}^{rs} (U_i][(V_i) = A \otimes B^t , \tag{5.3}$$

which obtains if, and only if,

$$\sum_{i=1}^{rs} U_i \otimes \overline{V}_i = (A][B^*) \ . \qquad (5.4)$$

Moreover,

$$V_i = A^*\hat{U}_i B^* \ , \quad i = 1 \ , \ 2 \ ,\ldots, \ rs \qquad (5.5)$$

Proof. Display (5.3) follows from (5.1) with $A \otimes B^t$ replacing $\underset{\sim}{R}$, while (5.4) follows from (5.3) by using the the duality Theorem 4.2. The exact form (5.5) for the V_i's follows from (5.2). (We note, an explicit description for the V_i's is missing from [2, Th. 3.4].) We observe that for the special case $\underset{\sim}{R} = A \otimes B^t$, we have $rg\underset{\sim}{R} = \mathcal{L}(rgB^*, rgA)$, and $rg\underset{\sim}{R}^* = \mathcal{L}(rgB, rgA^*)$ (cf. (2.5)) so that the hypotheses of Theorem 5.2 are consistent with those of Theorem 5.1; and the proof is done.□

The case $\underset{\sim}{R} = 0$.

We raise the question: When do dyads or (with help of Th. 4.2) Kronecker products sum to zero? We remark that an answer to this question is an answer to the question: When do dyads (resp. Kronecker products) sum to another dyad (resp. Kronecker product)? This is clear since

$$\sum_{i=1}^{N} (U_i][V_i) = 0 \rightleftarrows \sum_{i=1}^{N-1} (U_i][(V_i) = -(U_N][V_N) \ .$$

For the statement of the next theorem, we will not need the fact that the compenent vectors U_i , V_i are linear transformations. Accordingly, they will be presented only as elements in Hilbert space. The proof of the following theorem is direct, given Theorem 3.2.

Theorem 5.3. Given linearly independent $\{U_1, U_2, \ldots, U_N\}$

in Hilbert space H , and k vectors

$$U_{r+t} = \sum_{i=1}^{r} \alpha_i^{(r+t)} U_i \in sp\langle U_i : i = 1,2,\ldots,r\rangle$$

$$t = 1\ ,\ 2\ ,\ldots,\ k\ .$$

Then for any k-element set of vectors

$$V_{r+1}\ ,\ V_{r+2}\ ,\ldots,\ V_{r+k} \in H\ ,$$

there is one and only one r-element set of vectors

$$V_1\ ,\ V_2\ ,\ldots,\ V_r \in H \tag{5.6}$$

such that

$$\sum_{i=1}^{r+k} (U_i][V_i) = 0\ , \tag{5.7}$$

which holds true if, and only if,

$$\sum_{i=1}^{r+k} U_i \otimes \overline{V}_i = 0\ . \tag{5.8}$$

Moreover, the V_i of (5.6), (5.7) and (5.8) are given by

$$V_i = -\sum_{j=r}^{r+k} \overline{\alpha}_i^{(j)} V_j\ ,\quad i = 1\ ,\ 2\ ,\ldots,\ r\ . \tag{5.9}$$

The case $\underset{\sim}{R} = \underset{\sim}{I}$, the identity.

It is known, of course, that if $\sum_{i=1}^{N} (U_i][V_i) = I_N$, the identity in N-dimensional space, then it suffices that $U_i = V_i$ and $\{U_i\}$ is an orthonormal basis. Slightly more general is the condition that $\{U_i\}$ and $\{V_i\}$ be complementary biorthormal sets (cf. (3.3)). But Theorem 3.2 subsumes both these cases and presents the general situation as follows:

<u>Theorem 5.4</u> (decompositions of the identity). Given any spanning set $U_1\ ,\ U_2\ ,\ldots,\ U_N$ for Hilbert space H , i.e.,

$sp\langle U_i\rangle = H$, where dimension $H = r \le N$. Suppose the first r vectors U_1 , U_2 ,..., U_r for a basis for H . Then for $n - r$ arbitrary vectors

$$V_{r+1} \, , \; V_{r+2} \, , \ldots , \; V_N \in H \, ,$$

there is a unique r-element set of vectors

$$V_1 \, , \; V_2 \, , \ldots , \; V_r \in H \tag{5.10}$$

such that

$$\sum_{i=1}^{N} (U_i][V_i) = I_r \, , \text{ the identity on } H \, .$$

Moreover, the vectors of (5.10) are given by

$$V_i = \hat{U}_i - \sum_{j=r+1}^{N} \bar{\alpha}_i^{(j)} V_j \, , \quad i = 1 \, , \; 2 \, , \ldots , \; r \, , \tag{5.11}$$

where the scalars $\alpha_i^{(j)}$ are defined by

$$U_j = \sum_{i=1}^{r} \alpha_i^{(j)} U_i \, , \quad j = r+1 \, , \; r+2 \, , \ldots , \; N \, .$$

<u>Remark</u>. The context of Theorem 5.4 does not lend itself to an interesting dual statement via Theorem 4.2, unless we are interested in when the sum of Kronecker products $U_i \otimes \overline{V}_i$ produces the particular rank one operator $\phi(I_r):z \to zI_r$, for all complex z .

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